

Dynamical cascade generation as a basic mechanism of Benjamin-Feir instability

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A novel model of discretized energy cascade generated by Benjamin-Feir instability is presented. Conditions for appearance of direct and inverse cascades are given explicitly, as well as conditions for stabilization of the wave system due to cascade termination. These results can be used directly for explanation of available results of laboratory experiments and as basic forecast scenarios for planned experiments, depending on the frequency of an initially excited mode and steepness of its amplitude.

1. INTRODUCTION

Benjamin-Feir instability (BF-instability) is one of fundamental principles of nonlinear water wave dynamics, [1]. This phenomenon is of the utmost importance for description of dynamics and downshifting of energy spectrum among sea surface waves, formation of freak (or giant) waves in oceans and wave breaking. BF-instability and its physical applications have been profoundly studied during the last few decades, [2–5], and its main features can be briefly summarized as follows:

I. initial exponential growth of the main side bands of the carrier wave;

II. late asymmetry of sidebands and temporary frequency downshifting;

III. discretized spreading of energy to higher and lower frequencies;

IV. existence or absence of near recurrence Fermi-Pasta-Ulam (FPU) phenomenon in no breaking regime of wave propagation, in experiments with different parameters of initial excitation.

As for surface gravity waves on deep water resonant interactions occur at the third order, BF-instability can be described at early stages of the process as interaction of three monochromatic wave trains: carrier (ω_c), upper ($\omega_+ = \omega_c + \Delta\omega$) and lower ($\omega_- = \omega_c - \Delta\omega$) side-band waves with $\Delta\omega > 0$ which form a resonant quartet for one particular configuration which occurs when two of the waves coincide, with frequency resonance condition

$$\omega_+ + \omega_- = 2\omega_c. \quad (1)$$

where $\omega_i \equiv \omega(\mathbf{k}_i)$ and \mathbf{k}_i are notations for dispersion function and wave vector correspondingly.

In this Letter we present a novel model of BF-instability basing on the model of dynamical cascade generation in a wave system with narrow initial excitation, first introduced in [6]. We demonstrate that in the frame of our model a) main features **I–IV**, of BF-instability are naturally reproduced; and b) dependence of the cascade form on details of initial excitation (choice of frequency and wave steepness) is in accordance with available experimental data.

2. DYNAMICAL EQUATIONS

Dynamical system corresponding to an isolated quartet reads:

$$\begin{cases} i \dot{A}_1 = T A_2^* A_3 A_4 + (\tilde{\omega}_1 - \omega_1) A_1, \\ i \dot{A}_2 = T A_1^* A_3 A_4 + (\tilde{\omega}_2 - \omega_2) A_2, \\ i \dot{A}_3 = T^* A_4^* A_1 A_2 + (\tilde{\omega}_3 - \omega_3) A_3, \\ i \dot{A}_4 = T^* A_3^* A_1 A_2 + (\tilde{\omega}_4 - \omega_4) A_4, \\ \tilde{\omega}_j - \omega_j = \sum_{i=1}^4 (T_{ij} |A_j|^2 - \frac{1}{2} T_{jj} |A_i|^2), \end{cases} \quad (2)$$

where interaction coefficients $T_{ij} = T_{ji} \equiv T_{ij}^{ij}$ and $T = T_{34}^{12}$ are responsible for the nonlinear shifts of frequency and the energy exchange within a quartet correspondingly; $(\tilde{\omega}_j - \omega_j)$ are Stokes-corrected frequencies and A_j are slowly changing amplitudes of resonant modes in canonical variables. Explicit form of interaction coefficients for surface gravity waves is given in [16]. Analytical solution of (2) in terms of elliptic functions is found in [7] and is studied numerically.

Aiming to analyze quartet dynamics *qualitatively* in the case when only one or two modes are initially excited, one can use standard change of variables $A_j \equiv -iC_j \exp[-i\varphi_j]$, and rewrite (2) in amplitude-phase presentation as

$$\begin{aligned} \frac{dC_1^2}{dt} &= \frac{dC_2^2}{dt} = -\frac{dC_3^2}{dt} = -\frac{dC_4^2}{dt} \\ &= 2|T|C_1C_2C_3C_4 \sin(\arg T - \varphi_{12,34}), \end{aligned} \quad (3)$$

where the dynamical phase $\varphi_{12,34} \equiv \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4$ corresponds to the chosen resonance conditions of the form

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \quad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4. \quad (4)$$

Sys.(3) has three independent Manley-Rowe constants of motion

$$I_{13} = C_1^2 + C_3^2, I_{14} = C_1^2 + C_4^2, I_{23} = C_2^2 + C_3^2. \quad (5)$$

Any linear combinations of these three are also constants of motion, e.g.

$$I_{24} = C_2^2 + C_4^2, \quad I_{1234} = C_1^2 + C_2^2 + C_3^2 + C_4^2, \quad (6)$$

$$I_{1,2} = C_1^2 - C_2^2, \quad I_{3,4} = C_3^2 - C_4^2 \quad (7)$$

and can be used for qualitative analysis of a quartet dynamics for specific initial conditions.

Case 1: one mode is initially excited. Let initially the mode ω_1 be excited, i.e. at the time moment $t = 0$ we have

$$C_{1,0} \gg C_{2,0} \simeq C_{3,0} \simeq C_{4,0} \equiv c_0, \quad (8)$$

where $C_{j,0} \equiv C_j(t = 0)$. It follows from (5), that the amplitudes $C_2(t)$, $C_3(t)$ and $C_4(t)$, being initially small, remains small at all time moments $t > 0$. Indeed,

$$2c_0^2 \sim I_{24} + I_{14} - I_{3,4} = C_2^2 + C_3^2 \Rightarrow C_2(t) \sim C_3(t) \sim c_0, \quad (9)$$

$$0 \sim I_{13} - I_{14} = C_3^2 - C_4^2 \Rightarrow C_4(t) \sim c_0, \quad (10)$$

for any $t > 0$, i.e. the energy transfer in this case is essentially suppressed.

Case 2: two modes are initially excited. There exist two different types of modes' pairs which should be regarded separately, [6]: one-side-pair and two-side-pair, referring to resonance conditions of the form (4). Accordingly, there are two 1-pairs of modes' frequencies:

$$1\text{-pairs: } (\omega_1, \omega_2), (\omega_3, \omega_4), \quad (11)$$

and four 2-pairs of modes' frequencies:

$$2\text{-pairs: } (\omega_1, \omega_3), (\omega_1, \omega_4), (\omega_2, \omega_3), (\omega_4, \omega_3). \quad (12)$$

Case 2a: 2-pair is initially excited. Let

$$C_{1,0} \simeq C_{3,0} \gg C_{2,0} \simeq C_{4,0} \equiv c_0, \quad (13)$$

then again it follows from the form of Manley-Rowe constants of motion that $C_2(t) \sim C_4(t) \sim c_0$.

Case 2b: 1-pair is initially excited. Regard 1-pair (ω_1, ω_2) with initial modes' amplitudes as follows:

$$C_{1,0} \sim C_{2,0} \gg C_{3,0} \simeq C_{4,0}. \quad (14)$$

There exists no restriction on modes' growth originating from the Manley-Rowe constants of motion, and resulting evolution depends on the details of the initial energy distribution within a quartet.

During initial evolution, during which inequalities (14) still hold, one can neglect the effect of modes ω_3, ω_4 on modes ω_1, ω_2 and Eqs. (2) can be solved explicitly:

$$\begin{cases} A_1(t) = C_{1,0} \exp(i\Delta_{1,0}t), \\ A_2(t) = C_{2,0} \exp(i\Delta_{2,0}t), \\ A_3(t) = C_{3,0} \exp[(i\tilde{\omega}_3 + \nu_{12})t], \\ A_4^*(t) = C_{4,0} \exp[(-i\tilde{\omega}_4 + \nu_{12})t], \end{cases} \quad (15)$$

where

$$\begin{aligned} \Delta_{1,0} &= \frac{T_{11}}{2}C_{1,0}^2 + T_{12}C_{2,0}^2, \\ \Delta_{2,0} &= \frac{T_{22}}{2}C_{2,0}^2 + T_{12}C_{1,0}^2, \\ \Delta_{3,0} &= T_{31}C_{1,0}^2 + T_{32}C_{2,0}^2, \\ \Delta_{4,0} &= T_{41}C_{1,0}^2 + T_{42}C_{2,0}^2, \\ \tilde{\omega}_3 &= (\Delta_{1,0} + \Delta_{2,0} - \Delta_{3,0} + \Delta_{4,0})/2, \\ \tilde{\omega}_4 &= (\Delta_{1,0} + \Delta_{2,0} + \Delta_{3,0} - \Delta_{4,0})/2, \\ \nu_{12}^2 &= |\mathcal{P}|^2 - \frac{1}{4} \left(\sum_{j=1}^4 \Delta_{j,0} \right)^2 \\ &= |T|^2 C_{1,0}^2 C_{2,0}^2 - (\mathcal{T}_1 C_{1,0}^2 + \mathcal{T}_2 C_{2,0}^2)^2 / 4, \\ \mathcal{P} &= T^* C_{1,0} C_{2,0} \exp[i(\Delta_{1,0} + \Delta_{2,0})t], \\ \mathcal{T}_1 &= \frac{1}{2}T_{11} + T_{12} + T_{13} + T_{14}, \\ \mathcal{T}_2 &= T_{12} + \frac{1}{2}T_{22} + T_{23} + T_{24}. \end{aligned} \quad (16)$$

Accordingly, the evolution of the amplitudes A_3 and A_4 is defined by the sign of the increment ν_{12}^2 given by (16).

If $\nu_{12}^2 > 0$, (16) predicts exponential grow of amplitudes A_3 and A_4 . In this case energy goes from the initially excited 1-pair (ω_1, ω_2) to the second 1-pair (ω_3, ω_4) with characteristic time $\simeq 1/\nu_{12}$. Similarly, if the 1-pair (ω_3, ω_4) is initially excited, energy can effectively go to the 1-pair (ω_1, ω_2) , if $\nu_{34} > 0$.

However, (16) does not guarantee that ν_{12}^2 is positive. For instance, if $C_{1,0} \gg C_{2,0}$ or $C_{1,0} \ll C_{2,0}$, ν_{12}^2 is negative for any interaction coefficient while in the case $C_{1,0} = C_{2,0}$ the sign of ν_{12}^2 depends on the relations between $|T|$ and $(\mathcal{T}_1 + \mathcal{T}_2)/2$:

$$\nu_{12}^2 = [|T|^2 - (\mathcal{T}_1 + \mathcal{T}_2)^2 / 4] C_{1,0}^4, \quad (17)$$

If $\nu_{12}^2 < 0$, the solution (15) yields pure oscillatory behavior of the amplitudes A_3, A_4 with frequencies $\tilde{\omega}_3 \pm |\nu_{12}|$ and $\tilde{\omega}_4 \pm |\nu_{12}|$ correspondingly.

3. GENERATION OF A CASCADE

The general model of dynamical cascade generation in 3- and 4-wave systems with narrow initial excitation is sketched in [6]. In this Letter we work out the details and apply it for the description of BF-instability.

At the initial step, $n = 0$, frequency resonance conditions have the same form (1) for both direct and inverse cascade while the only form of a quartet in which *one-mode excitation* yields generation of resonant interactions reads

$$\omega_1 + \omega_2 = 2\omega_3 \quad (18)$$

and only in the case when the mode with frequency ω_3 is excited. This occurs due to Hasselmann's criterion of instability for 4-wave systems, [8]. For all other configurations and initial excitations, any single mode in a quartet is neutrally stable. The most effective resonance takes place if (18) is satisfied exactly, i.e. $\Delta\omega$ should be the same for both ω_+ and ω_- .

Hasselmann's criterion can be applied at each further step yielding the general form (18) at each step of a cascade. To simplify further presentation we introduce notation $\omega_{\pm n} = \omega_c \pm n\Delta\omega$.

At the step 1, a couple of new modes ω_1 and ω_{-1} is generated. From pure kinematical considerations, at the next steps of a cascade all possible quartets of the form

$$\omega_c \pm \omega_1 \mp \omega_{-1} = \omega_c \pm 2\Delta\omega \quad (19)$$

are possible. However, different dynamical properties of 1-pair and 2-pair in a quartet define two possible combinations only, yielding exact frequency resonance condition for quartets of the form (18) *with excited 1-pair of modes*:

$$\begin{aligned} \omega_c + \omega_c + 2\Delta\omega &= 2(\omega_c + \Delta\omega) \\ \Rightarrow \omega_c + \omega_2 &= 2\omega_1, \end{aligned} \quad (20)$$

$$\begin{aligned} \omega_c + \omega_c - 2\Delta\omega &= 2(\omega_c - \Delta\omega) \\ \Rightarrow \omega_c + \omega_{-2} &= 2\omega_{-1}. \end{aligned} \quad (21)$$

Accordingly, the beginning of direct and inverse cascades is given by (20) and (21). As $\omega_c + \omega_{\pm 2} = 2\omega_{\pm 1}$, frequency shift $|\omega_c - \omega_{\pm 2}| = 2\Delta\omega$ occurs (its sign is opposite for direct and inverse cascade).

Similarly, at the step n we have

$$\begin{aligned} \omega_c + \omega_c + 2n\Delta\omega &= 2(\omega_c + n\Delta\omega) \\ \Rightarrow \omega_{n+1} + \omega_{n-1} &= 2\omega_n \end{aligned} \quad (22)$$

$$\begin{aligned} \omega_c + \omega_c - 2n\Delta\omega &= 2(\omega_c - n\Delta\omega) \\ \Rightarrow \omega_{-(n+1)} + \omega_{-(n-1)} &= 2\omega_{-n}, \end{aligned} \quad (23)$$

and the frequency shift is again $2\Delta\omega$, i.e. its magnitude is the same at each step of both direct and inverse cascade. The complete system describing both direct and inverse cascade takes now the following form:

$$\begin{cases} \omega_+ + \omega_- = 2\omega_c \\ \omega_c + \omega_{\pm 2} = 2\omega_{\pm 1} \\ \omega_{\pm 1} + \omega_{\pm 3} = 2\omega_{\pm 2} \\ \dots \\ \omega_{\pm(n+1)} + \omega_{\pm(n-1)} = 2\omega_{\pm n}. \end{cases} \quad (24)$$

Direction of cascade depends on the sign chosen in the lower index $\omega_{\pm n}$. Indeed, the choice of ω_{-n} generates a sequence of frequencies $\omega_c > \omega_{-1} > \dots > \omega_{-n}$, i.e. inverse cascade, while the choice of ω_{+n} yields direct cascade $\omega_c < \omega_1 < \dots < \omega_n$.

4. DYNAMICS OF CASCADE

Exponential growth of the two main side-bands ω_1, ω_{-1} at the initial step $n = 0$ is caused by the resonance (18) at the expense of the initially excited mode $\omega_3 = \omega_c$ and has non-dimensional increment (in physical variables, i.e. $I \sim \nu_{12}$)

$$0 < I = \varepsilon^2 \sigma (2 - \sigma^2)^{1/2} / 2 < 1, \quad (25)$$

where $\sigma = \Delta\omega / \varepsilon\omega_c$ is the ratio of the relative resonance frequency bandwidth of the wave train to the initially excited wave steepness $\varepsilon = C_c k_c$.

Maximal growth rate is the same for sub- and super-harmonics of the excited wave:

$$I_{max} = \varepsilon^2 / 2 = C_c^2 k_c^2 / 2 \quad (26)$$

At the next step $n = 1$ direct and inverse cascade will be initialized by two already exciting modes ω_1 and ω_{-1} correspondingly.

However, the important observation is that at the first step of a cascade the maximum of the increments for direct $I_{dir,+}$ and inverse $I_{inv,-}$ cascades differ:

$$I_{dir,+} \sim C_+^2 k_+^2 / 2 \quad I_{inv,-} \sim C_-^2 k_-^2 / 2. \quad (27)$$

This means that in the regime of symmetrical growth of the main side-bands $C_+ \sim C_-$, the *asymmetrical behavior of the growing modes* will be observed since

$$k_+ > k_- \Rightarrow I_{dir,+} > I_{inv,-}. \quad (28)$$

Similar considerations allow us to conclude that at the n -th step of the cascade again

$$I_{dir,+n} > I_{inv,-n}. \quad (29)$$

The last equation shows that at each step, increment of instability for direct cascade is bigger than increment of instability for inverse cascade.

This clearly explains the *frequency downshift phenomenon* – the main higher side-band component spreads energy to higher frequency modes

$$\omega_c < \omega_1 < \dots < \omega_n$$

by direct cascade mechanism much faster than the main lower side-band by the inverse cascade with corresponding sequence of frequencies

$$\omega_c > \omega_{-1} > \dots > \omega_{-n}.$$

Accordingly, main lower side-band mode finally will be dominant which is manifestation of frequency downshift phenomenon.

This scenario can be clearly observed e.g. in [5] where laboratory observations of wave group evolution, including breaking effects are presented. In particular, it is

stated there (p. 223) that "the initial distribution of energy (...) is altered by the transfer of energy to free waves $\omega_0 \pm \delta\omega$, first noticeably to $n = +2$, then $n = -2, +3$, etc. These relatively fast transfers seems to be a consequence of detuned resonances of which first is

$$\begin{cases} (\omega_0) \mp (\omega_0 - \delta\omega) \pm (\omega_0 + \delta\omega) = \omega_0 \pm 2\delta\omega, \\ (k_0) \mp (k_0 - \delta k) \pm (k_0 + \delta k) = k_0 \pm 2\delta k + \Delta k, \end{cases} \quad (30)$$

in which Δk is a small detuning factor." As in our notations $\omega_0 = \omega_c$ and $\delta\omega = \Delta\omega$, the frequency resonance condition above is just a more compact form of two frequency resonance conditions (20),(21) for quartets of the form (18) with excited 1-pair of modes.

Experiments presented in [5] have been performed in a large wave tank (50 x 4.2 x 2.1 m), for the range of carrier wave initial steepness $0.12 < \varepsilon < 0.28$ and wavelengths 1 to 4 m. Both direct and inverse discretized energy spectra have been clearly observed (see e.g. Fig.20 from [5]), breaking and recurrence phenomena have been reproduced.

The authors concluded that "a serious challenge is imposed by these results to any predictive method of ocean wave evolution in which downshifting depends solely on slow, high-order, resonant wave interactions" which "might very well dominate the downshifting of energetic waves in the ocean, too" ([5], p.226).

We conclude here that Tulin and Waseda in [5] came very close to the conclusive solution of this problem while describing the first few steps of dynamical cascade. Only one final step – realization that corresponding detuned resonances build clusters of the form (24) – has not been done.

5. TERMINATION OF CASCADE

Energy cascading will be terminated as soon as the BF-criterion of instability (25) is violated, i.e. if $I \approx 0$ (stabilization of the wave system due to transition to linear regime) and if $I > 1$ (substantial increasing of the cascading modes' steepness and transition to breaking regime). In the last case

$$\sigma^2 = (\Delta\omega/\omega_c/\varepsilon)^2 > 2. \quad (31)$$

For the n -th step of cascade, the magnitude of σ^2 can be estimated as follows:

$$\sigma^2 = \frac{(\Delta\omega)^2}{(\omega_n\varepsilon)^2} = \frac{(\Delta\omega)^2}{(\omega_c + n\Delta\omega)^2 C_0^2 (k_c + n\Delta k)^2} > 2. \quad (32)$$

We can obtain upper estimate of the energy spreading at the n -th step of the cascade in the resonance of form (18) by its maximal value, when the excited carrier wave energy will be totally and equally distributed between two resonance side-band modes:

$$C_n^2 = C_{n-1}^2 + C_{n+1}^2 \sim 2C_{n+1}^2 \Rightarrow C_{n+1}^2 \leq C_n^2/2.$$

This means that *the general upper estimate* for the n -th step of cascade has the form

$$C_n^2 \leq C_0^2/2^n. \quad (33)$$

After substituting (33) to the condition (32), this yields

$$\sigma^2 = \frac{(\Delta\omega)^2}{(\omega_n\varepsilon)^2} = \frac{2^n(\Delta\omega)^2}{(\omega_c + n\Delta\omega)^2 C_0^2 (k_c + n\Delta k)^2} > 2. \quad (34)$$

The latter inequality shows clearly that the stability criterion will be reached after *a finite number of steps* for the direct energy cascade; correspondingly, energy spreading to higher frequencies will be terminated due to the growth of nonlinearity and consequent breaking effects which is in accordance with laboratory results [5].

For the inverse cascade we can estimate the stabilization condition in a similar manner as

$$\sigma^2 = \frac{(\Delta\omega)^2}{(\omega_{-n}\varepsilon)^2} = \frac{2^n(\Delta\omega)^2}{(\omega_c - n\Delta\omega)^2 C_0^2 (k_c - n\Delta k)^2} > 2, \quad (35)$$

i.e. the inverse cascade will be terminated even faster than the direct cascade.

These estimates have been made in accordance with classical results (25),(26) and are valid only for small enough initial steepness of the excited wave $\varepsilon \sim 0.1$.

For moderate and high initial wave train steepness $\varepsilon \sim 0.1 \div 0.4$, improved results for the growth of increments and a criterion of instability have been obtained by Dysthe, [9]. In this case, instability violates if

$$\sigma^2 > 2\left(1 - \frac{2\Delta\omega}{\omega_c}\right) \quad (36)$$

This obviously means that termination of energy cascades can happen even earlier for a higher initial steepness of carrier wave.

As it is shown in [9], maximal growth rate is modified by the following way (see Eq.(3.10), p. 111, notations as in the cited paper):

$$\gamma_m = \frac{1}{2}(1 - 2A_0)A_0^2$$

where $A_0 = ak$ is steepness of carrier wave. Accordingly, in our notations

$$I_{max} = \frac{1}{2}\varepsilon^2(1 - 2\varepsilon) = \frac{1}{2}C_c^2 k_c^2(1 - 2C_c k_c). \quad (37)$$

Expressions for $I_{dir,+n}$ and $I_{inv,-n}$ at the first cascade's step read

$$I_{dir,+} = \frac{1}{2}C_+^2 k_+^2(1 - 2C_+ k_+), \quad k_+ = k_c + \Delta k, \quad (38)$$

$$I_{dir,-} = \frac{1}{2}C_-^2 k_-^2(1 - 2C_- k_-), \quad k_- = k_c - \Delta k, \quad (39)$$

and in the regime of symmetrical growth of the main side-bands $C_+ \sim C_- \sim C$ we have

$$\begin{aligned}
I_{\text{dir},+} - I_{\text{dir},-} &\sim \\
&\frac{1}{2}C^2[k_+^2(1 - 2Ck_+) - k_-^2(1 - 2Ck_-)] \\
&= \frac{1}{2}C^2[(k_+^2 - k_-^2) - 2C(k_+^3 - k_-^3)] \\
&= \frac{1}{2}C^2(k_+ - k_-)[k_+ + k_- - 2C(k_+^2 + k_+k_- + k_-^2)] \\
&= 2C^2\Delta k[k_c - C(3k_c^2 + (\Delta k)^2)]. \quad (40)
\end{aligned}$$

As $C, k, \Delta k$ are positive, it follows from (40) that $I_{\text{dir},+} > I_{\text{dir},-}$ only if the following condition holds

$$k_c - C(3k_c^2 + (\Delta k)^2) > 0, \quad (41)$$

which is satisfied in the finite range of wavelengths k_c . Similar considerations show that at n -th cascade's step, $I_{\text{dir},+n} > I_{\text{dir},-n}$ is also satisfied in the finite range of k_c such that $k_c - C(3k_c^2 + (n\Delta k)^2) > 0$. For all other k_c , maximal increment for inverse cascade will be larger than for direct cascade, $I_{\text{inv},-n} > I_{\text{dir},+n}$.

Thus, by considering two forms of instability increment given by (26) and (37) for the energy cascade concept, we can make the conclusion that for essentially high initial steepness $\varepsilon \gtrsim 0.25$ the inverse dynamical energy cascade to lower frequencies may dominate. In general, both directions of cascade may be significant, unlike the case of a small initial steepness of the excited wave.

6. CONCLUSIONS

- The main features **I-IV**, formulated in the Introduction have been previously studied mainly numerically, while our model gives a clear explanation of the *physical origin of the observed phenomena*.

- Moreover, several fundamental facets of BF-instability that have not been investigated theoretically until now – cascade direction, its dependence on the initial conditions and finiteness of the number of cascading modes – are adequately and constructively described by our model:

- Direction of cascade depends on the initial wave steepness of the wave train; for small enough steepness $\varepsilon \sim 0.1$ direct energy cascade prevails, while for high enough steepness $\varepsilon \gtrsim 0.25$ inverse cascade and energy spreading to lower frequency modes may be comparable with direct cascade rates.

- An increase of initial steepness may reduce the number of steps for energy cascades in both directions.

- Frequency downshifting in the wave system is caused by the direct energy cascade and spreading of energy to higher frequencies at the expense of the main super side-band mode; this process dominates for small to moderate initial steepness of the excited waves.

- Inverse cascade can be expected to be the prevailing process for high enough basic carrier wave steepness.

- Only a finite number of steps in both direct and inverse energy cascade leading to both breaking or stabilization regimes satisfy the BF-criterion.

- Fermi-Pasta-Ulam recurrent process may occur if at some cascade step modes are generated which are in *exact* resonance, i.e. $\Delta k = 0$ in (30) (this is only *necessary condition*). This general prediction is in accordance with experimental results where it was shown that the observed recurrence is very close to the NLS or three-wave system of the form (18) and "neither a reduction in the number of waves per group nor downshifting of the spectral energy" is observed in the experiments of Tulin and Waseda ([5], p. 210) conducted in the laboratory tank of the sizes 50 x 4.2 x 2.1 m.

The evolution of wave trains on an effectively much longer fetch than previous studies (330 x 5 x 5 m) was experimentally investigated in [3, 4] with following results. A very long scaled wave modulation with several modulation loops demonstrates re-stabilization processes: finally the periodic modulations of wave train are observed at the latest stage with the most energetic lower side-band wave. This means the final termination of the wave cascade mechanism and stabilization of the system with essentially large steepness in accordance with our model.

For small initial steepness of the wave packet $\varepsilon \sim 0.1$ Benjamin-Feir instability leads to growing of the main pair side-band modes. Asymmetrical growth of side-bands with prevailing of the lower side-band mode and discretized energy cascade to the higher frequencies was also observed in the experiments [3, 4]. Near recurrence FPU phenomenon was clearly seen at the latest stages of the wave propagation: most part of wave energy revert back to the carrier frequency mode. For large enough initial steepness of the wave packet $\varepsilon \sim 0.15$ to 0.25 wave steepness during propagation leads to wave breaking. Periodic modulation and demodulation of wave trains are found at post-breaking stage, in which the energy of wave train transfers between the carrier wave and a pair of sidebands. Only partial FPU phenomenon was observed in this case - essential part of energy is lost due to breaking.

Necessary and sufficient conditions for manifestation of this phenomenon depend explicitly on the relation between wavelength and initial phase of the carrier wave and the aspect ratio of the laboratory tank (see (17) and remark afterward; more details can be found in [6]). Deduction of their explicit form is outside the scope of this Letter where we concentrated on the study of non-periodic energy transfer due to the novel mechanism - dynamic cascade given by (24).

- Our model of BF-instability can be further refined in many aspects. For instance, dissipation can easily be included at each step by changing of C_n to $p_n C_n$ in (31), with some constants p_n , $0 < p_n < 1$. Accordingly, the

form of the increments $I_{\text{dir},+n}, I_{\text{inv},-n}$ will be modified and the number of cascade's steps will be reduced. Detailed study of the effect of dissipation can be found in [11].

• This model is quite general and describes basic energy cascade in *an arbitrary 4-wave system with narrow initial excitation* though the estimate (25) and definition of the small parameter ε might change, depending on the specifics of the wave system. In particular, the model can be used directly as a basic description of wind generated instabilities of surface water waves from which extreme (or rogue) waves originate, e.g. see [12] for numerical simulations and [13, 14] for laboratory study. Theoretical study of the role of Benjamin-Feir instability in formation of extreme waves can be found e.g. in [15].

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